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Interpolation and bipolar ordered structures

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Abstract

In decision making, capacities (monotone set functions) and the Choquet integral have been recently generalized to the framework of bicapacities, which are monotone two-places set functions, where the first argument is the subset related to positive outcomes, and the second argument the subset related to negative outcomes. Bicapacities can be thought as a bipolarization of capacities. We show that this construction can be done in a very general way. First we reconsider capacities and the Choquet integral through the notions of geometric realization of a distributive lattice and its natural triangulation. Second, we propose a general mechanism of bipolarization of a given structure, and its geometric realization and natural triangulation.

Keywords: interpolation, Choquet integral, lattice, bipolar structure

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1 Introduction

Our work is motivated by two facts, which have their roots mainly in decision theory.

The first concerns the Choquet integral with respect to a capacity, a well-known tool in decision under uncertainty [16] and multicriteria decision making [11]. It has been remarked by Grabisch [7] that the Choquet integral is the simplest linear interpolator between vertices of the hypercube $[0,1]^n$. This interpolation was first introduced by Lovász [13], and Singer [17].

The second fact concerns bipolarity in decision making. Bipolarity is inherent to the affect (see, e.g., Slovic et al. [18]), and roughly speaking it concerns the distinction between good and bad outcomes, which are not treated symmetrically in decision behaviour. This has given rise to the well-known Cumulative Prospect Theory of Tversky and Kahnemann [20], where the underlying model (called hereafter CPT) is a difference of two Choquet integrals, one for good outcomes and the other for bad outcomes. Recently, the authors have proposed the notion of bicapacity and its associated Choquet integral [9, 10], which takes fully into account the bipolarity, and includes as a particular case the CPT model.

Our aim is to study how bipolarity can be put into a given structure, and how to extend functions, operations, transforms (like the Choquet integral, the Möbius transform, etc.) on the bipolarized structure. To motivate this, we go more into mathematical details, and take the example of capacities and bicapacities, the latter being a bipolarization of the former (necessary definitions are all given in Section 2).

Let us consider as starting point capacities defined on some given finite set $N := \{1, \dots, n\}$. i.e., mappings $\nu : 2^N \to [0,1]$, such that $\nu(\varnothing) = 0$ and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A natural structure for this definition is to take the Boolean lattice $(2^N, \subseteq)$, so that a capacity can be seen as an isotone mapping from $(2^N, \subseteq)$ to $([0,1], \leq)$, preserving the bottom element.

With this structure, the Möbius transform m of a capacity ν is the solution of the equation

$$\nu(A) = \sum_{B \subseteq A} m(B), \quad \forall A \subseteq N.$$

It is well known that the Choquet integral w.r.t ν of some function f on N is given in terms of the Möbius transform by

$$\int f \, d\nu = \sum_{A \subseteq N} m(A) \bigwedge_{i \in A} f(i), \tag{1}$$

an expression which is particularly simple.

Let us now turn to the case of bicapacities, which are mappings $v : \mathcal{Q}(N) \to [-1,1]$ defined on $\mathcal{Q}(N) := \{(A,B) \in 2^N \times 2^N \mid A \cap B = \varnothing\}$ such that $v(\varnothing, \varnothing) = 0$, and $v(A,B) \leq v(C,D)$ whenever $(A,B), (C,D) \in \mathcal{Q}(N)$ with $A \subseteq C$ and $B \supseteq D$. A natural structure for $\mathcal{Q}(N)$ is to consider the partial order $\sqsubseteq$ defined by

$$(A, A') \sqsubseteq (B, B') \Leftrightarrow A \subseteq B \text{ and } A' \supseteq B'.$$

Then $(\mathcal{Q}(N), \sqsubseteq)$ is the lattice 3^n, with top element $(N, \varnothing)$, bottom element $(\varnothing, N)$, and supremum and infimum given by

$$(A, A') \sqcap (B, B') = (A \cap B, A' \cup B'), \quad (A, A') \sqcup (B, B') = (A \cup B, A' \cap B').$$

Doing so, bicapacities are isotone functions from $(\mathcal{Q}(N), \sqsubseteq)$ to $([-1,1], \leq)$ preserving the bottom elements, so that the situation is very similar to classical capacities. Fig. 1 shows $(\mathcal{Q}(N), \sqsubseteq)$ for $n = 2$ (12 is a shorthand for $\{1,2\}$, etc.). The Möbius transform m of a bicapacity v is then

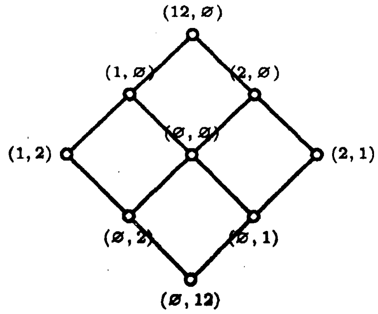

Figure 1: The lattice $(\mathcal{Q}(N), \sqsubseteq)$ with $n = 2$

determined by solving the following equation:

$$v(A, A') = \sum_{(B,B')\sqsubseteq(A,A')} m(B, B').$$

Its solution is given by [9]:

$$m(A, A') = \sum_{\substack{(B,B')\sqsubseteq(A,A')\\ B'\cap A=\varnothing}} (-1)^{|A\setminus B|+|B'\setminus A'|} v(B, B') = \sum_{\substack{B\subseteq A\\ A'\subseteq B'\subseteq A^c}} (-1)^{|A\setminus B|+|B'\setminus A'|} v(B, B').$$

In [10], it is shown that the Choquet integral w.r.t. a bicapacity v expressed in terms of the Möbius transform writes:

$$\int f\,dv = \sum_{B\subseteq N} m(\varnothing, B)\Big(\bigwedge_{i\in B^c\cap N^-} f(i)\Big) + \sum_{\substack{(A,B)\in\mathcal{Q}(N)\\ A\neq\varnothing}} m(A, B)\Big[\Big(\bigwedge_{i\in(A\cup B)^c\cap N^-} f(i) + \bigwedge_{i\in A} f(i)\Big)\vee 0\Big].$$

This complicated expression, which contrasts with (1), questions the validity of the structure $(\mathcal{Q}(N), \sqsubseteq)$. In fact, the Choquet integral w.r.t. a bicapacity is constructed by a symmetry around the $(\varnothing, \varnothing)$ point, which is the "center" point in $\mathcal{Q}(N)$ (see Def. 4). On the other hand, the lattice $(\mathcal{Q}(N), \sqsubseteq)$ has no center, but only a top and a bottom element, and in particular the Möbius transform by its definition is rooted at the bottom point.

This suggests to take a different order on $\mathcal{Q}(N)$, which is simply the product order:

$$(A, A') \subseteq (B, B') \Leftrightarrow A \subseteq B \text{ and } A' \subseteq B'.$$

This order was chosen by Bilbao in his first works on bicooperative games [2]. The structure we obtain is illustrated on Fig. 2. We see that $(\mathcal{Q}(N), \subseteq)$ is an inf-semilattice (in fact, the restriction of $2^N \times 2^N$ endowed with the product order to $\mathcal{Q}(N)$). Indeed, $(A, B) \wedge (C, D) = (A\cap C, B\cap D)$ always exists in $\mathcal{Q}(N)$ since $(A \cap C) \cap (B \cap D) = (A \cap B) \cap (C \cap D) = \varnothing$. On the contrary $(A, B) \vee (C, D)$ does not exist in general. It is much clearer to redraw this figure exactly as Fig. 1, and indicating by arrows the order relation (see Fig. 3). This shows clearly the construction of the bipolar structure: the original unipolar structure (top, in white) is duplicated and reversed (bottom, in grey), then combinations of positive and negative elements complete the structure (left and right, crossed). Let us call it the *bipolar extension* of 2^N. Observe the symmetry of arrows w.r.t. the horizontal line passing through the central point.

Let us define the Möbius transform with this structure, which we denote by b. For any

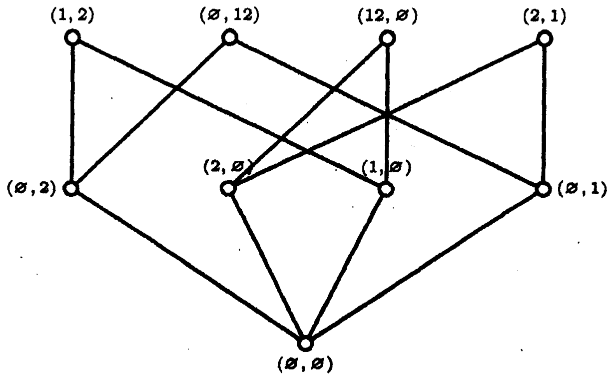

Figure 2: $(\mathcal{Q}(N), \subseteq)$ with $n = 2$

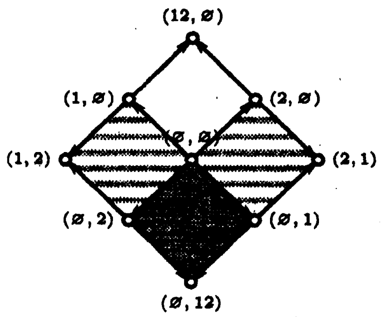

Figure 3: The bipolar extension of 2^N with $n = 2$

bicapacity v, b is the solution of:

$$v(A_1, A_2) = \sum_{(B_1,B_2)\subseteq(A_1,A_2)} b(B_1, B_2) = \sum_{\substack{B_1\subseteq A_1\\ B_2\subseteq A_2}} b(B_1, B_2).$$

This gives:

$$b(A_1, A_2) = \sum_{\substack{B_1\subseteq A_1\\ B_2\subseteq A_2}} (-1)^{|A_1\setminus B_1|+|A_2\setminus B_2|} v(B_1, B_2).$$

This expression was suggested directly (without consideration of some order structure) by Fujimoto [5]. The expression of the Choquet integral writes in a much more elegant way, close to the case of capacities (compare with (1)):

$$\int f\,dv = \sum_{(A_1,A_2)\in\mathcal{Q}(N)} b(A_1, A_2)\Big[\bigwedge_{i\in A_1} f^+(i) \wedge \bigwedge_{j\in A_2} f^-(j)\Big], \tag{2}$$

with $f^+ = f \vee 0$, $f^- = (-f)^+$, showing that the structure is much more adequate.

The aim of our work is to provide a general construction for the bipolar extension of some ordered structure, and then to define on it any functional obtained by interpolation (as the Choquet integral) by a replication of some original functional. This work has been inspired essentially by Koshevoy, who used the geometric realization of a lattice and its natural triangulation [12], and by Fujimoto [6], who first remarked the inadequacy of our original definition of the Möbius transform for bicapacities.

2 Preliminaries

In this section, we consider a finite index set $N := \{1, \dots, n\}$.

2.1 The Möbius transform

We recall from Rota [15] the following facts about the Möbius transform. Let us consider f, g two real-valued functions on a locally finite poset $(X, \leq)$ with bottom element, such that

$$g(x) = \sum_{y \leq x} f(y). \tag{3}$$

The solution of this equation in term of g is given through the Möbius function μ by

$$f(x) = \sum_{y \leq x} \mu(y, x) g(y) \tag{4}$$

where μ is defined inductively by

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq t < y} \mu(x, t), & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

Note that μ depends only on the structure of $(X, \leq)$. Function f is called the *Möbius transform* (or *inverse*) of g.

2.2 Capacities

Definition 1 *(i) A function $\nu : 2^N \rightarrow \mathbb{R}$ is a* game *if it satisfies $\nu(\varnothing) = 0$.*

(ii) A game which satisfies $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$ (monotonicity) is called a capacity *[4] or* fuzzy measure *[19]. The capacity is* normalized *if in addition $\nu(N) = 1$.*

Unanimity games are capacities of the type

$$u_A(B) := \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{else} \end{cases}$$

for some $A \subseteq N, A \neq \varnothing$.

Definition 2 *Let us consider $f : N \rightarrow \mathbb{R}_+$. The* Choquet integral *of f w.r.t. a capacity ν is given by*

$$\int f \, d\nu := \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i+1))] \nu(\{\pi(1), \dots, \pi(i)\}),$$

where π is a permutation on N such that $f(\pi(1)) \geq \dots \geq f(\pi(n))$, and $f(\pi(n+1)) := 0$.

The above definition is valid if ν is a game. For any $\{0, 1\}$-valued capacity ν on 2^N we have (see, e.g., [14]):

$$\int f \, d\nu = \bigvee_{A | \nu(A) = 1} \bigwedge_{i \in A} f(i). \tag{5}$$

We can apply the general definition of the Möbius transform to capacities: when $(X, \leq) = (2^N, \subseteq)$, it is well known that the Möbius function becomes, for any $A, B \in 2^N$

$$\mu(A,B) = \begin{cases} (-1)^{|B\setminus A|} & \text{if } A \subseteq B \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

If g is some capacity ν, then its Möbius transform m is, using (4):

$$m(A) = \sum_{B \subseteq A} (-1)^{|A\setminus B|} \nu(B).$$

It is well known that the set of unanimity games is a basis for all games, whose coordinates in this basis are exactly the Möbius transform:

$$\nu = \sum_{A \subseteq N, A \neq \emptyset} m(A) u_A. \tag{7}$$

Note that (7) is just a rewriting of (3) for $(2^N, \subseteq)$. Equation (1) gives the expression of the Choquet integral w.r.t. the Möbius transform of ν.

2.3 Bicapacities

We introduce $\mathcal{Q}(N) := \{(A,B) \in 2^N \times 2^N \mid A \cap B = \emptyset\}$.

Definition 3 *(i) A mapping $v : \mathcal{Q}(N) \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$ is a* bicooperative game *[1].*

(ii) A bicooperative game v such that $v(A,B) \leq v(C,D)$ whenever $(A,B), (C,D) \in \mathcal{Q}(N)$ with $A \subseteq C$ and $B \supseteq D$ (monotonicity) is called a bicapacity *[8, 9]. Moreover, a bicapacity is* normalized *if in addition $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$.*

Definition 4 *Let v be a bicapacity and f be a real-valued function on N. The* (general) Choquet integral *of f w.r.t v is given by*

$$\int f\, dv := \int |f|\, d\nu_{N_f^+}$$

where $\nu_{N_f^+}$ is a game on N defined by

$$\nu_{N_f^+}(C) := v(C \cap N_f^+, C \cap N_f^-), \quad \forall C \subseteq N$$

and $N_f^+ := \{i \in N \mid f(i) \geq 0\}$, $N_f^- = N \setminus N_f^+$.

When there is no fear of ambiguity, we drop subscript f in N_f^+, N_f^-. Note that the definition remains valid if v is a bicooperative game.

Considering on $\mathcal{Q}(N)$ the product order

$$(A, A') \subseteq (B, B') \Leftrightarrow A \subseteq B \text{ and } A' \subseteq B',$$

the Möbius transform b of a bicapacity v is the solution of:

$$\begin{aligned} v(A_1, A_2) &= \sum_{(B_1,B_2) \subseteq (A_1,A_2)} b(B_1, B_2) \\ &= \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} b(B_1, B_2). \end{aligned}$$

This gives:

$$b(A_1, A_2) = \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} v(B_1, B_2).$$

Unanimity games are then naturally defined by

$$u_{(A_1,A_2)}(B_1, B_2) := \begin{cases} 1, & \text{if } (B_1, B_2) \supseteq (A_1, A_2) \\ 0, & \text{else.} \end{cases}$$

and form a basis of bicooperative games.

The expression of the Choquet integral in terms of b is given by (2).

3 Lattices, geometric realizations, and triangulation

A *lattice* is a set L endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \vee y$ and greatest lower bound $x \wedge y$ always exist. For finite lattices, the greatest element of L (denoted $\top$) and least element $\bot$ always exist. x *covers* y (denoted $x \succ y$) if $x > y$ and there is no z such that $x > z > y$. A sequence of elements $x \leq y \leq \cdots \leq z$ of L is called a *chain* from x to z, while an *antichain* is a sequence of elements such that it contains no pair of comparable elements. A chain from x to z is *maximal* if no element can be added in the chain, i.e., it has the form $x \prec y \prec \cdots \prec z$.

The lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently, j is join-irreducible if it covers only one element. Join-irreducible elements covering $\bot$ are called *atoms*, and the lattice is *atomistic* if all join-irreducible elements are atoms. The set of all join-irreducible elements of L is denoted $\mathcal{J}(L)$.

For any $x \in L$, we say that x *has a complement in* L if there exists $x' \in L$ such that $x \wedge x' = \bot$ and $x \vee x' = \top$. The complement is unique if the lattice is distributive.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way. We denote by $\eta(x)$ the *(normal) decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Hence

$$x = \bigvee_{j \in \eta(x)} j.$$

Note that this decomposition may be redundant. Let us rephrase differently the above result. We say that $Q \subseteq L$ is a *downset* of L if $x \in Q$, $y \in L$ and $y \leq x$ imply $y \in Q$. For any subset P of L, we denote by $\mathcal{O}(P)$ the set of all downsets of P. Then the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff's theorem [3]). Also,

$$\eta(x \vee y) = \eta(x) \cup \eta(y), \quad \eta(x \wedge y) = \eta(x) \cap \eta(y) \tag{8}$$

if L is distributive.

We introduce now the notion of *geometric realization* of a lattice, following Koshevoy [12]. A first fact to notice is that downsets of some partially ordered set P correspond bijectively to nonincreasing mappings from P to $\{0, 1\}$. Let us denote by $\mathcal{D}(P)$ the set of all nonincreasing mappings from P to $\{0, 1\}$. Then Birkhoff's theorem can be rephrased as follows: *any distributive lattice L is isomorphic to $\mathcal{D}(\mathcal{J}(L))$.*

Consider next for any partially ordered set P the set $\mathcal{C}(P)$ of nonincreasing mappings from P to $[0, 1]$. It can be easily shown that $\mathcal{C}(P)$ is a convex polyhedron, whose set of vertices is $\mathcal{D}(P)$.

Definition 5 *The* geometric realization *of a distributive lattice L is the set $\mathcal{C}(\mathcal{J}(L))$.*

Example: If L is the Boolean lattice 2^N, with $N := \{1, \dots, n\}$, then $\mathcal{J}(L) = N$ (atoms). We have $\mathcal{D}(\mathcal{J}(L)) = \{x : N \to \{0,1\}, x \text{ nonincreasing}\}$, but since N is an antichain, there is no restriction on x and $\mathcal{D}(\mathcal{J}(L)) = \{0,1\}^N$, i.e., it is the set of vertices of $[0,1]^n$. Similarly, $\mathcal{C}(\mathcal{J}(L)) = [0,1]^N$, which is the hypercube itself. □

REMARK 1:

(i) From this example, it is important to note that for a distributive lattice with n join-irreducible elements, the elements of L (more precisely, of $\mathcal{D}(\mathcal{J}(L))$) can be seen as a subset of the vertices of $[0,1]^n$, and the convex hull of this subset of vertices is precisely the geometric realization of L. For a given element x of L, $\eta(x)$ is the set of coordinates in $[0,1]^n$ being equal to 1, the others being 0. Hence, by (8), the subset of vertices is closed under coordinatewise minimum and maximum (corresponding respectively to infimum and supremum of L), and it always contains $(0, \dots, 0)$ and $(1, \dots, 1)$.

(ii) So far, we have seen three equivalent ways of representing a distributive lattice L: the set of downsets $\mathcal{O}(\mathcal{J}(L))$ of its join-irreducible elements, the nonincreasing functions in $\mathcal{D}(\mathcal{J}(L))$, and a subset of vertices of $[0,1]^{|\mathcal{J}(L)|}$. Specifically:

$$x \in L \leftrightarrow \eta(x) \in \mathcal{O}(\mathcal{J}(L)) \leftrightarrow 1_{\eta(x)} \in \mathcal{D}(\mathcal{J}(L)) \leftrightarrow (1_{\eta(x)}, 0_{\eta(x)^c}) \in [0,1]^{|\mathcal{J}(L)|} \tag{9}$$

where the notation $(1_A, 0_{A^c})$ denotes a vector whose coordinates are 1 if in A, and 0 otherwise. All arrows represent isomorphisms, the leftmost one being an isomorphism if L is distributive.

Let us now introduce the *natural triangulation* of $\mathcal{C}(\mathcal{J}(L))$, following Koshevoy again. It consists in partitioning $\mathcal{C}(\mathcal{J}(L))$ into simplices whose vertices are in $\mathcal{D}(\mathcal{J}(L))$. To each chain in $\mathcal{D}(\mathcal{J}(L))$, say $C := \{1_{X_0} < 1_{X_1} < \cdots < 1_{X_p}\}$ where the X_i's are downsets of $\mathcal{J}(L)$, corresponds a p-dimensional simplex $\sigma(C) := \mathrm{co}(1_{X_0}, 1_{X_1}, \dots, 1_{X_p})$. It can be shown that these simplices cover $\mathcal{C}(\mathcal{J}(L))$ such that any f in $\mathcal{C}(\mathcal{J}(L))$ belongs to the interior of a unique simplex. Any f in $\sigma(C)$ writes

$$f = \sum_{i=0}^{p} \alpha_i 1_{X_i}, \quad \sum_{i=0}^{p} \alpha_i = 1, \quad \alpha_i \geq 0, \forall i. \tag{10}$$

Then f has value 1 on X_0, value $1 - \alpha_0$ on $X_1 \setminus X_0$, value $1 - \alpha_0 - \alpha_1$ on $X_2 \setminus X_1$, etc., and value α_p on $X_p \setminus X_{p-1}$. Note that by definition of downsets f is nonincreasing.

Example (ctd): Let us take $L = 2^N$, and consider a maximal chain in $\mathcal{D}(\mathcal{J}(L))$, denoted by $C := \{1_{A_0} < 1_{A_1} < \cdots < 1_{A_n}\}$, and $\emptyset =: A_0 \subset A_1 \subset \cdots \subset A_n := N$. For each such maximal chain (thus defining a n-dimensional simplex), there exists a permutation π on N such that $A_i = \{\pi(1), \dots, \pi(i)\}$. Since $1_\emptyset \equiv 0$, we have for any $f \in \sigma(C)$:

$$f(j) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(j) = \sum_{i | j \in A_i} \alpha_i, \quad \forall j \in N.$$

Observe that $f(\pi(1)) = 1 - \alpha_0$, $f(\pi(n)) = \alpha_n$, and in general $f(\pi(i)) = 1 - \sum_{j=0}^{i-1} \alpha_j$, $i \in N$. Moreover, there are $n!$ n-dimensional simplices. □

REMARK 2: The above example shows that the case of a maximal chain from bottom to top of L is particular and of special interest. Specifically, assume as in Remark 1 that L is distributive, let $\mathcal{J}(L) := \{1, \ldots, n\}$, and take any maximal chain $C := \{1_\varnothing = 0 \prec 1_{X_1} \prec \cdots \prec 1_{X_{|\mathcal{J}(L)|}} = 1\}$. Then the simplex $\sigma(C)$ is n-dimensional, it contains vertices $(0, \ldots, 0)$ and $(1, \ldots, 1)$ in $[0,1]^n$, and the sequence $X_0, \ldots, X_n$ defines a permutation π on $\mathcal{J}(L)$ such that $X_i = \{\pi(1), \ldots, \pi(i)\}$, $i = 1, \ldots, n$. Hence

$$f(j) = \sum_{i=1}^{n} \alpha_i 1_{X_i}(j) = \sum_{X_i \ni j} \alpha_i = \sum_{i=\pi^{-1}(j)}^{n} \alpha_i, \quad j = 1, \ldots, n. \tag{11}$$

Inverting this triangular system, one immediately obtains

$$\alpha_i = f(\pi(i)) - f(\pi(i+1)), \quad i = 1, \ldots, n-1, \text{ and } \alpha_n = f(\pi(n)) \tag{12}$$

and $\alpha_0 = 1 - \sum_{i=1}^{n} \alpha_i = 1 - f(\pi(1))$. Note that $f(\pi(1)) \geq f(\pi(2)) \geq \cdots \geq f(\pi(n))$. Lastly, remark that any f belongs to such a n-dimensional simplex (but not necessarily in the interior), so that formulas (11) and (12) can always be used.

4 Natural interpolative functions

Let us consider a distributive lattice L, and any real-valued function F on L (or on $\mathcal{D}(\mathcal{J}(L))$). An interesting question is how to extend this function to the geometric realization of L. Infinitely many extensions exist, but the above material on triangulation brings us a very simple answer to this question. Remark that (10) expresses any point of some simplex of the geometric realization $\mathcal{C}(\mathcal{J}(L))$ as a convex combination with at most $n+1$ points of $\mathcal{D}(\mathcal{J}(L))$. Hence, the extension $\overline{F}$ of F over this simplex of $\mathcal{C}(\mathcal{J}(L))$ could be taken as the same convex combination of values of F at vertices of the simplex. This leads to the following definition.

Definition 6 *For any functional $F : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$ on a distributive lattice L, its* natural extension to the geometric realization of L *is defined by:*

$$\overline{F}(f) := \sum_{i=0}^{p} \alpha_i F(1_{X_i})$$

for all $f \in \mathrm{int}(\sigma(C))$, with C being a chain $\{1_{X_0} < 1_{X_1} < \cdots < 1_{X_p}\}$ in $\mathcal{D}(\mathcal{J}(L))$, and $\sigma(C)$ its convex hull in $\mathcal{C}(\mathcal{J}(L))$, with $f = \sum_{i=0}^{p} \alpha_i 1_{X_i}$.

Example (ctd): Consider again $L = 2^N$, and take the notations introduced before in this example. Using (12), we get

$$\begin{aligned}\overline{F}(f) &= \sum_{i=1}^{n} \alpha_i F(1_{A_i}) \\ &= \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i+1))] F(1_{\{\pi(1), \ldots, \pi(i)\}}),\end{aligned}$$

with the convention $f(\pi(n+1)) := 0$. Putting $\mu(A) := F(1_A)$, we recognize the Choquet integral $\int f \, d\nu$ (see Definition 2). □

This example shows that the Choquet integral is the natural extension of capacities.

REMARK 3:

(i) Using Remark 2, we can always write f under the form (11), so that using (12), we have as in the example before

$$\overline{F}(f) = \sum_{i=1}^{n}[f(\pi(i)) - f(\pi(i+1))]F(1_{\{\pi(1),\ldots,\pi(i)\}}) \tag{13}$$

with $n := |\mathcal{J}(L)|$ and $f(\pi(n+1)) := 0$. By analogy, this could be called the *Choquet integral w.r.t. F*. Moreover, using Remark 1, we could consider F as a game or capacity defined over a sublattice of the Boolean lattice 2^n.

(ii) It follows from the definition of $\overline{F}$ and (12) that $\overline{F}$ is linear in each simplex $\sigma(C)$, i.e., $\overline{F}(f+g) = \overline{F}(f)+\overline{F}(g)$ provided that $f, g, f+g$ belongs to the same $\sigma(C)$. Moreover, $\overline{F}$ is linear in F, in the sense that $\overline{F+G}(f) = \overline{F}(f) + \overline{G}(f)$ for any f.

(iii) This extension can be seen as a *parsimonious linear interpolation* since it linearly interpolates on the vertices of the geometric realization, using the less possible number of vertices. Of course, the natural triangulation is not the only one decomposing a convex polyhedron into simplices, hence other parsimonious linear interpolation can be defined as well.

5 Bipolar structures

5.1 Bipolar extension of L

Definition 7 *Let us consider $(L, \leq)$ an inf-semilattice with bottom element $\perp$. The* bipolar extension *$\widetilde{L}$ of L is defined as follows:*

$$\widetilde{L} := \{(x, y) \mid x, y \in L, x \wedge y = \perp\},$$

which we endow with the product order $\leq$ on L^2.

Remark that $\widetilde{L}$ is a downset of L^2. The following holds.

Proposition 1 *Let $(L, \leq)$ be an inf-semilattice.*

(i) $(\widetilde{L}, \leq)$ is an inf-semilattice whose bottom element is $(\perp, \perp)$, where $\leq$ is the product order on L^2.

(ii) The set of join-irreducible elements of $\widetilde{L}$ is

$$\mathcal{J}(\widetilde{L}) = \{(j, \perp) \mid j \in \mathcal{J}(L)\} \cup \{(\perp, j) \mid j \in \mathcal{J}(L)\}.$$

(iii) The normal decomposition writes

$$(x, y) = \bigvee_{j \leq x, j \in \mathcal{J}(L)} (j, \perp) \vee \bigvee_{j \leq y, j \in \mathcal{J}(L)} (\perp, j).$$

We consider now the Möbius function over $\widetilde{L}$. The aim is to solve

$$f(x, y) = \sum_{(x', y') \leq (x, y), (x', y') \in \widetilde{L}} g(x', y'), \quad \forall (x, y) \in \widetilde{L}, \tag{14}$$

where f, g are real-valued functions on $\widetilde{L}$. The solution is given through the Möbius function on $\widetilde{L}$:

$$g(x,y) = \sum_{\substack{(z,t)\leq(x,y) \\ (z,t)\in\widetilde{L}}} f(z,t)\mu_{\widetilde{L}}((z,t),(x,y)). \tag{15}$$

The following holds.

Proposition 2 *The Möbius function on $\widetilde{L}$ is given by:*

$$\mu_{\widetilde{L}}((z,t),(x,y)) = \mu_L(z,x)\mu_L(t,y).$$

Note that as usual, the set of functions $u_{(x,y)}$ defined by

$$u_{(x,y)}(z,t) = \begin{cases} 1, & \text{if } (z,t) \geq (x,y) \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

forms a basis of the functions on $\widetilde{L}$.

Theorem 1 *Let L be a finite distributive lattice, and $c(L)$ be the set of its complemented elements. Then, for any $x \in c(L)$, its complement being denoted by x', the interval $L(x)$ of $\widetilde{L}$ defined by*

$$L(x) := [(\bot,\bot),(x,x')]$$

and endowed with the product order of L^2 is isomorphic to L, by the order isomorphism $\phi_x : L(x) \rightarrow L$, $(y,z) \mapsto y \vee z$. The inverse function ϕ_x^{-1} is given by $\phi_x^{-1}(w) = (w \wedge x, w \wedge x')$.

Moreover, the join-irreducible elements of $L(x)$ are the image of those of L by ϕ_x^{-1}, i.e.:

$$\mathcal{J}(L(x)) = \{(j \wedge x, j \wedge x') \mid j \in \mathcal{J}(L)\}.$$

Remark that in any finite lattice, $\bot$ and $\top$ are complemented elements, and $L(\top) = L$, $L(\bot) = L^*$, where L^* is the dual of L. An interesting question is whether the union of all $L(x)$, $x \in c(L)$, is equal to $\widetilde{L}$.

Theorem 2 *Let L be a finite distributive lattice. Then the bipolar extension $\widetilde{L}$ can be written as:*

$$\widetilde{L} = \bigcup_{x\in c(L)} L(x)$$

if and only if $\mathcal{J}(L)$ has all its connected components with a single bottom element.

Example (ctd): Consider $L = 2^N$. Then $\widetilde{L} = \mathcal{Q}(N)$. Since 2^N is Boolean, any element $A \subseteq N$ is complemented ($A' = A^c$), and $2^N(A) = [(\varnothing,\varnothing),(A,A^c)]$. Obviously the conditions of Theorem 2 are satisfied, thus

$$\mathcal{Q}(N) = \bigcup_{A\subseteq N} [(\varnothing,\varnothing),(A,A^c)].$$

Remark 4: This important result shows that $\widetilde{L}$ is composed by "tiles", all identical to L, as in Fig. 3. Hence, we call a *regular mosaic* any $\widetilde{L}$ satisfying conditions of Theorem 2. There are two important particular cases of regular mosaics:

(i) L is a product of m linear lattices (totally ordered). Then

$$c(L) = \{(\top_A, \bot_{A^c}) \mid A \subseteq \{1, \dots, m\}\}$$

where $(\top_A, \bot_{A^c})$ has coordinate number i equal to $\top_i$ if $i \in A$, and $\bot_i$ otherwise. Also, $(\top_A, \bot_{A^c})' = (\bot_A, \top_{A^c})$. This case covers Boolean lattices (case of capacities), and lattices of the form k^m.

(ii) $\mathcal{J}(L)$ has a single connected component with one bottom element. Then $\widetilde{L}$ contains only elements of the form $(y, \bot)$ or $(\bot, z)$, i.e., $\widetilde{L} = L(\bot) \cup L(\top)$.

5.2 Bipolar geometric realization

Since $\widetilde{L}$ is not a distributive lattice, it is not possible to define its geometric realization in the sense of Def. 5. Assuming that $\widetilde{L}$ is a regular mosaic, we propose the following definition.

Definition 8 *Let $\widetilde{L}$ be a regular mosaic, and $x \in c(L)$. We consider the mappings $\xi_x : \mathcal{J}(L) \to \{-1, 0, 1\}$ such that*

(i) $|\xi_x|$ is nonincreasing

(ii) $\xi_x(j) \geq 0$ if $j \in \eta(x)$

(iii) $\xi_x(j) \leq 0$ if $j \in \eta(x')$.

The set of such functions is denoted by $\mathcal{D}_x(\mathcal{J}(L))$. Similarly, we introduce

$$\mathcal{C}_x(\mathcal{J}(L)) := \{f_x : \mathcal{J}(L) \to [-1, 1] \text{ such that } |f_x| \text{ is nonincreasing}, \\ f_x(j) \geq 0 \text{ if } j \in \eta(x), f_x(j) \leq 0 \text{ if } j \in \eta(x')\}. \quad (17)$$

Then the bipolar geometric realization of L *is*

$$|\widetilde{L}| := \bigcup_{x \in c(L)} \mathcal{C}_x(\mathcal{J}(L)).$$

Proposition 3 *For any $x \in c(L)$, $\mathcal{D}_x(\mathcal{J}(L))$ is the set of vertices of $\mathcal{C}_x(\mathcal{J}(L))$.*

Proposition 4 *Let $x \in c(L)$. There is a bijection $\psi_x : \mathcal{D}_x(\mathcal{J}(L)) \to L(x)$ defined by $\psi_x(\xi) := (y_\xi, z_\xi)$ with*

$$\eta(y_\xi) = \{j \in \mathcal{J}(L) \mid \xi(j) = 1\}, \quad \eta(z_\xi) = \{j \in \mathcal{J}(L) \mid \xi(j) = -1\}, \quad (18)$$

and the inverse function is defined by $\psi_x^{-1}(y, z) := \xi_{(y,z)}$ with

$$\xi_{(y,z)}(j) := \begin{cases} 1, & \text{if } j \in \eta(y) \\ -1, & \text{if } j \in \eta(z) \\ 0, & \text{otherwise,} \end{cases} \quad (19)$$

for any $j \in \mathcal{J}(L)$, or in more compact form

$$\xi_{(y,z)} = 1_{\eta(y)} - 1_{\eta(z)}.$$

Example (ctd): Consider $L = 2^N$, and some $N^+ \subseteq N$, $N^- := N \setminus N^+$. Then

$$\mathcal{D}_{N^+}(N) = \{\xi_{N^+} : N \to \{-1, 0, 1\} \text{ such that } (\xi_{N^+})_{|N^+} \geq 0, \quad (\xi_{N^+})_{|N^-} \leq 0\}.$$

Moreover, $\psi_{N^+}(\xi_{N^+}) = (\{j \in N \mid \xi_{N^+}(j) = 1\}, \{j \in N \mid \xi_{N^+}(j) = -1\})$.

REMARK 5: Observe that functions $\xi_x \in \mathcal{D}_x(\mathcal{J}(L))$ corresponds to a subset of points of $[-1,1]^{|\mathcal{J}(L)|}$ of the form $(1_A, (-1)_B, 0_{(A\cup B)^c})$, with $A \subseteq \eta(x)$ and $B \subseteq \eta(x')$, and that $\mathcal{C}_x(\mathcal{J}(L))$ is the convex hull of these points.

We end this section by addressing the natural triangulation of the bipolar geometric realization. Let us consider some f in $\mathcal{C}(\mathcal{J}(L))$, assuming $f = \sum_{i=0}^p \alpha_i 1_{X_i}$, with $1_{X_0}, \ldots, 1_{X_p}$ forming a chain in $\mathcal{D}(\mathcal{J}(L))$. Given $x \in c(L)$, let us define the corresponding f_x in $\mathcal{C}_x(\mathcal{J}(L))$ as follows:

$$\begin{aligned} f_x &:= \sum_{i=0}^p \alpha_i \psi_x^{-1}(\phi_x^{-1}(\eta^{-1}(X_i))) \\ &= \sum_{i=0}^p \alpha_i (1_{X_i \cap \eta(x)} - 1_{X_i \cap \eta(x')}). \end{aligned}$$

Explicitely, this gives, for any $j \in \mathcal{J}(L)$:

$$f_x(j) = \begin{cases} \sum_{i|j\in X_i} \alpha_i, & \text{if } j \in \eta(x) \\ -\sum_{i|j\in X_i} \alpha_i, & \text{if } j \in \eta(x'). \end{cases}$$

Hence $|f_x|$ takes value 1 on X_0, $1 - \alpha_0$ on $X_1 \setminus X_0$, etc., and is nonincreasing.

Remark that $|f_x| = f$ if $f \in \mathcal{C}(\mathcal{J}(L))$, and $|f|_x = f$ if $f \in \mathcal{C}_x(\mathcal{J}(L))$.

5.3 Natural interpolation on bipolar structures

Assume $F : \bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L)) \to \mathbb{R}$ is given. We want to define the extension $\overline{F}$ of this functional on the bipolar geometric realization $|\widetilde{L}|$.

Let us take $f \in |\widetilde{L}| = \bigcup_{x\in c(L)} \mathcal{C}_x(\mathcal{J}(L))$. First, we must choose $x \in c(L)$ such that f belongs to $\mathcal{C}_x(\mathcal{J}(L))$ (x is not unique in general). Defining

$$\mathcal{J}(L)^+ := \{j \in \mathcal{J}(L) \mid f(j) \geq 0\}, \quad \mathcal{J}(L)^- := \mathcal{J}(L) \setminus \mathcal{J}(L)^+,$$

it suffices to take x, x' defined by

$$\eta(x) := \bigcup_{k\in K} J_k, \quad \eta(x') := \mathcal{J}(L) \setminus \eta(x)$$

with K the smallest one such that $\mathcal{J}(L)^+ \subseteq \bigcup_{k\in K} J_k$ (using notations of proof of Theorem 2). Now, consider $|f|$, which belongs to $\mathcal{C}(\mathcal{J}(L))$, and its expression using the natural triangulation:

$$|f| = \sum_{i=0}^p \alpha_i 1_{X_i}$$

with $1_{X_0}, \ldots, 1_{X_p}$ a chain in $\mathcal{D}(\mathcal{J}(L))$. Then we have $|f|_x = f$, and we propose the following definition.

Definition 9 *Assume $\widetilde{L}$ is a regular mosaic. For any functional $F : \bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L)) \to \mathbb{R}$, its* natural extension to the bipolar geometric realization *of $\widetilde{L}$ is defined by:*

$$\overline{F}(f) := \sum_{i=0}^{p} \alpha_i F_x(1_{X_i})$$

for all $f \in C_x(\mathcal{J}(L))$, letting $|f| := \sum_{i=0}^{p} \alpha_i 1_{X_i}$ for some chain $\{1_{X_0} < 1_{X_1} < \cdots < 1_{X_p}\}$ in $\mathcal{D}(\mathcal{J}(L))$, and $F_x : \mathcal{D}(\mathcal{J}(L)) \to \mathbb{R}$ defined by:

$$F_x(1_{X_i}) := F(1_{X_i\cap\eta(x)} - 1_{X_i\cap\eta(x')}).$$

Example (end): Let us take once more $L = 2^N$. For a given f, we define $N^+ := \{j \in N \mid f(j) \geq 0\}$ and $N^- := N \setminus N^+$, we have:

$$\overline{F}(f) = \sum_{i=1}^{n} \alpha_i F_{N^+}(1_{X_i}) = \sum_{i=1}^{n} \left[|f(\pi(i))| - |f(\pi(i+1))|\right] F(1_{X_i\cap N^+} - 1_{X_i\cap N^-}),$$

where we have used (12). Putting $v(A, B) := F(1_A - 1_B)$, we recognize the Choquet integral for bicapacities. □

REMARK 6: Definition 9 can be written equivalently as $\overline{F}(f) = \overline{F_x}(|f|)$, making clear the relation between the functional on L and on $\widetilde{L}$.

Lastly, we address the problem of expressing $\overline{F}$ in terms of the Möbius transform of F, using Prop. 2. For this purpose, it is better to turn a given functional F on $\bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L))$ into its equivalent form $\widetilde{F}$ defined on $\widetilde{L}$, thanks to the mappings ψ_x, $x \in c(L)$. Doing so, we can use Prop. 2 and (15), and get the Möbius transform of $\widetilde{F}$, which we denote by $\widetilde{m}$:

$$\widetilde{m}(x, y) = \sum_{\substack{(z,t)\leq(x,y)\\ (z,t)\in\widetilde{L}}} \widetilde{F}(z, t)\mu_L(z, x)\mu_L(t, y), \quad \forall(x, y) \in \widetilde{L}.$$

We need the following result, which is a generalization of (5).

Lemma 1 *Let $f \in C(\mathcal{J}(L))$ and $F : \mathcal{D}(\mathcal{J}(L)) \to \{0,1\}$ being nondecreasing and 0-1 valued. Then*

$$\overline{F}(f) = \bigvee_{\substack{T\subseteq\mathcal{J}(L)\\ F(1_T)=1}} \bigwedge_{j\in T} f(j).$$

The following is a generalization of (2).

Proposition 5 *With the above notations, for any $f \in |\widetilde{L}|$ and any F on $\bigcup_{x\in c(L)} \mathcal{D}_x(\mathcal{J}(L))$, the following holds:*

$$\overline{F}(f) = \sum_{(s,t)\in\widetilde{L}} \widetilde{m}(s, t)\Big[\bigwedge_{j\in\eta(s)} f^+(j) \wedge \bigwedge_{j\in\eta(t)} f^-(j)\Big],$$

with $f^+ = f \vee 0$, $f^- = (-f)^+$.

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